A Classification of General (2, 3) Point Correspondences Between Two Planes.

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§ 1. Introduction and General Discussion.

1. *Introduction.*—The purpose of this paper is to discuss and classify algebraic (2, 3) point correspondences between two planes. Such a correspondence is established between two planes (x) and (x') when an algebraic relation in x_1, x_2, x_3 and x'_1, x'_2, x'_3 exists such that to any point of (x) shall correspond two points of (x'), and to any point of (x') three points of (x).

While the theory of (1, 1) correspondences has been thoroughly developed and that of $(1, n)$ correspondences treated in some detail, there are but few writings that consider the case in which both planes are multiple and thus have a direct bearing on the subject of the present paper. The first paper on (2, 2) point correspondences was published in 1889 by P. Visalli.* In this paper he discusses a very special case in which the lines of either plane correspond to conics in the other. Burali-Forti † later obtained certain (2, 2) correspondences by combining two (1, 2) correspondences and showed that the case treated by Visalli is included in these. The first investigation of general (2, 2) point correspondences was made by G. Marletta.‡ He considers two special types and deduces the properties of their associated transformations geometrically in four dimensional space. Finally an exhaustive classification of (2, 2) point correspondences has been recently made by F. R. Sharpe and Virgil Snyder.§

The only paper to date on correspondences of multiplicities greater than two is that of Richard Baldus, ‖ who has investigated certain properties of (m_1, m_2) point correspondences by geometrical methods.

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* P. Visalli, "La trasformazione quadratica (2, 2)," *Rend. del Circ. Mat. di Palermo*, Vol. III (1889), pp. 165–170.

† C. Burali-Forti, "Sulle trasformazione (2, 2) che si possono ottenere mediante due trasformazioni doppie," *Rend. del Circ. Mat. di Palermo*, Vol, V (1891), pp. 91–99.

‡ G. Marletta, "La trasformazione quadratica (2, 2) fra piani," *Rend. del Circ. Mat. di Palermo*, Vol. XVII (1903), pp. 173–184. "La trasformazioni cubiche (2, 2) fra piani," same volume, pp. 371–385.

§ F. R. Sharpe and V. Snyder, "Types of (2, 2) Point Correspondences Between two Planes." *T. A. M. S.*, Vol. XVIII (1917), pp. 402–414.

‖ R. Baldus, "Zur Theorie der gegenzeitig mehrdeutigen algebraischen Ebenentransformationen," *Mat. Annal.*, Vol. LXXII (1912), pp. 1–56.

2. A general (2, 3) point correspondence is said to exist between two planes (x) and (x') when the points of the two planes are related as follows: Choosing (x) and (x') respectively for the double plane and the triple plane, to a point P' of (x') correspond three points P_1, P_2, P_3 of (x), ordinarily distinct. To P_1, P_2, P_3 correspond in (x') the original point P' and the residual points P_1', P_2', P_3', respectively, which are usually distinct from each other and from P'. One more step will fully illustrate the general procedure. To P_1', P_2', P_3', respectively, correspond the three sets P_1, P_4, P_5; P_2, P_6, P_7; P_3, P_8, P_9, the three points in each set being usually non-coincident.

If in (x') the three residual points P_1', P_2', P_3' coincide in one point P_0', then the three points P_1, P_2, P_3 of (x) correspond to either P' or P_0' of (x') and to each of the points P_1, P_2, P_3 corresponds the pair P', P_0'. Such an involutorial (2, 3) point correspondence may be obtained by combining a (1, 2) and a (1, 3) point correspondence and is known as a (2, 3) compound involution. Since (2, 3) compound involutions, although distinct types, are not general (2, 3) point correspondences, they will not be discussed in this paper.

In the general (2, 3) point correspondence between two planes, a type is distinguished and defined by two equations, each algebraic in the coordinates of both planes. As shown in the next article, such a pair of equations establishes an algebraic relationship between the planes (x) and (x') such that to a point of (x) correspond two points of (x'), and to a point of (x') correspond three points of (x). By analogy to Riemann surfaces, (x) is called the double plane and (x') the triple plane. By considering all the possible forms of these defining equations that do not establish (2, 3) compound involutions, twelve independent types of general (2, 3) point correspondences have been obtained. By independent types we mean those that can not be reduced, one to the other, by any series of birational transformations. It will be shown finally that all general (2, 3) point correspondences are birationally equivalent to some one of these twelve types.

3. *General Properties.*—The defining equations of a (2, 3) point correspondence between two planes (x) and (x') may be written as two algebraic equations of the form,

$$\Sigma u_k(x)\, u_k'(x') = 0, \tag{1}$$

$$\Sigma v_k(x)\, v_k'(x') = 0. \tag{2}$$

When x_1', x_2', x_3' are parameters, these equations represent two curves of (x) intersecting in three variable points corresponding to the point (x_1', x_2', x_3'),

and when x_1, x_2, x_3 are parameters, two curves of (x') intersecting in two variable points corresponding to the point (x_1, x_2, x_3). Connected with each (2, 3) point correspondence, the defining equations establish a transformation between the two planes (x) and (x') such that to a line $C_1(x)$ corresponds a curve $C'_n(x')$ of order n and genus p' which may have fundamental points A'_j of multiplicities r'_j. Likewise to a line $C'_1(x')$ corresponds a curve $C_n(x)$ of order n and genus p which may have fundamental points A_i of multiplicities r_i.

The fundamental or basis points of a given plane are fixed points on the curves of that plane. The order of a basis point is its multiplicity on the images of general lines. To each basis point of order r in either plane corresponds a basis curve of order r in the other plane. If a curve C passes through a basis point A its image is composite, consisting of the image of A counted as many times as the multiplicity of A on C and a residual curve called the proper image of C.

To a point P' of (x') correspond three points P_1, P_2, P_3 of (x) which describe the curve C_n as P' describes the line C'_1. Similarly, to a point P of (x) correspond two points P'_1, P'_2 of (x') which describe C'_n as P describes C_1. To the point of intersection of two lines of (x) [(x')] correspond in (x') [(x)] two [three] non-basic intersections of the image curves of these lines. These two [three] image points always lie at the two [three] intersections of the two curves of (x') [(x)] given by the defining equations in which the parameters are the coordinates of the common point of the two lines of (x) [(x')]. The images in both planes are distinct except for points on certain fixed curves.

If the two images P'_1, P'_2 of a point P of (x) coincide, P is on the branchpoint curve of (x) which will be called $L(x)$. The locus of the corresponding coincidences is the coincidence curve of (x'), denoted by $K'(x')$, which is in (1, 1) correspondence with $L(x)$. If in (x), two of the three images of the point P' of (x') coincide, P' is on the branchpoint curve $L'(x')$ and the coincident image points $P_1 \equiv P_2$ describe the coincidence curve $K(x)$ in (x) which is in (1, 1) correspondence with $L'(x')$. At the same time the remaining image point P_3 in (x) describes a fixed locus $\Gamma(x)$, the residual image of $L'(x')$ and also in (1, 1) correspondence with it. When all three image points coincide, $K(x)$ and $\Gamma(x)$ have a common tangent at $P_1 \equiv P_2 \equiv P_3$ and P' is a cusp on $L'(x')$. There are but a finite number of points of (x') whose three image points in (x) coincide unless every point of $L'(x')$ has this property.

The equation of $L(x)$ is the condition on the x_i that the two curves of (x') given by the defining equations be tangent, $L(x)$ has no non-basic multiple points. The proper image of $L(x)$ is $K'(x')$ counted twice. $K'(x')$ may have

non-basic multiple points since it is of the same genus as $L(x)$. In like manner the equation of $L'(x')$ is the condition on the x'_i that the two curves of (x) be tangent. Aside from the basis points, the only singularities of $L'(x')$ are cusps. The proper image of $L'(x')$ is $[K(x)]^2\Gamma(x)$. Both $K(x)$ and $\Gamma(x)$ are of the same genus as $L'(x')$ and may have non-basic multiple points. When the genus of the image curves of the straight lines of one plane is known, the order of the branchpoint curve of that plane can be readily found without the above-mentioned algebraic process which is always possible, but sometimes laborious. If t is the multiplicity of the transformation and p the genus, the number of coincidences of the transformation is given by the formula $2(t+p-1)$, known as "Zeuthen's formula."* Since these coincidences are in (1, 1) correspondence with the intersections of the branchpoint curve and a general line of the other plane, the order of that branchpoint curve is $2(t+p-1)$.

All image curves of (x) have only contacts with $L(x)$ in accordance with the lemma:

If a point P of (x) describe a curve C, the necessary and sufficient condition that its images P'_1, P'_2 of (x') describe distinct curves is that C touch $L(x)$ at every common point.

This lemma was proved for (2, 2) point correspondences† where it was valid for both planes. In the case of (2, 3) point correspondences, however, it holds only for the double plane (x). For the triple plane (x') we have the lemma:

If a point P' of (x') describe a curve C', the necessary and sufficient condition that its images P_1, P_2, P_3 of (x) describe distinct curves is that C' and $L'(x')$ have contacts and intersections respectively equal to the intersections of the image of C' with $K(x)$ and $\Gamma(x)$.

Applying these lemmas to the fixed curves of (x) and (x') we have the theorem:

In (x') $L'(x')$ and $K'(x')$ have r intersections and s tangencies corresponding in (x) to r tangencies of $L(x)$ and $\Gamma(x)$ and s tangencies of $L(x)$ and $K(x)$.

If a line C'_1 meets $K'(x')$ in i' points, its image in (x) is a curve C_n tangent to $L(x)$ at i' points. The image of C_n is C'_1 counted three times, and a

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* Zeuthen, "Nouvelle démonstration de théorèmes sur des séries de points correspondants sur deux courbes," *Mat. Annal.*, Vol. III (1871), p. 150.

† F. R. Sharpe and V. Snyder, *loc. cit.*, Article 2.

residual curve that cuts C'_1 in $2d+i'$ points corresponding to the d variable double points of C_n and the i' contacts of $L(x)$ and C_n. If a line C_1 meets $K(x)$ and $\Gamma(x)$ in i and j points respectively, its image in (x') is a curve C'_m which has i contacts and j intersections with $L'(x')$. The image of C'_n is C_1 counted twice, and a residual curve intersecting C_1 in $2d'+i$ points corresponding to the i contacts of C'_n and $L'(x')$ and the d' variable double points of C'_n.

4. *Types of Correspondences.*—There are twelve independent types of general (2, 3) point correspondences between two planes. Special cases may be common to two or more types. Each type is established by a set of defining equations as described in the preceding article. For convenience, these have been divided into three classes with respect to the curve system in the triple plane.

Class 1. Lines and conics; five types.

Class 2. Curves of order n having basis points of multiplicity $n-2$ at the vertex of the line pencil; four types.

Class 3. Conics with two basis points; three types.

The following table shows the curves employed in the defining equations for each type. The symbol $C_n; jP_i$ means a curve of order n with j basis points of each multiplicity i.

Class.	Type.	$u'_k(x')=0$.	$u_k(x)=0$.	$v'_k(x')=0$.	$v_k(x)=0$.
1	I	line	line	conic	cubic
	II	line	cubic	conic	line
	III	line	conic; P_1	conic	conic; P_1
	IV	line	cubic; $6P_1$	conic	cubic; $6P_1$
	V	line	conic; $2P_1$	conic	cubic; P_2P_1
2	VI	line pencil	cubic	$C_n; P_{n-2}$	line
	VII	line pencil	line pencil	$C_n; P_{n-2}$	$C_m; P_{m-3}$
	VIII	line pencil	cubic; $6P_1$	$C_n; P_{n-2}$	cubic; $6P_1$
	IX	line pencil	cubic; $8P_1$	$C_n; P_{n-2}$	$C_9; 8P_3$
3	X	conic; $2P_1$	line	conic; $2P_1$	cubic
	XI	conic; $2P_1$	conic; P_1	conic; $2P_1$	conic; P_1
	XII	conic; $2P_1$	cubic; $6P_1$	conic; $2P_1$	cubic; $6P_1$

Each of these types will now be discussed, Type I in detail and the others more briefly. Similar methods are used in investigating all the types, but the algebraic details naturally differ.

§ 2. CLASS 1. FIVE TYPES.

5. *Type I. Image of a Line.*—The defining equations,

$$\sum_{k=1}^{3} x_k x'_k = 0, \tag{1}$$

$$\sum_{k=1}^{6} u_k(x) v'_k(x') = 0, \tag{2}$$

where $u_k(x)=0$, $v'_k(x')=0$ are respectively cubics of (x) and conics of (x'), relate the point (x'_1, x'_2, x'_3) to the three intersections of the line and cubic determined by it in (x), and the point (x_1, x_2, x_3) to the two intersections of the line and conic determined by it in (x').

The image of a line, $C'_1 \equiv \sum_{k=1}^{3} a'_k x'_k = 0$, is the quintic

$$C_5 \equiv \sum_{k=1}^{6} u_k(x) v'_k(a'_2x_3 - a'_3x_2,\ a'_3x_1 - a'_1x_3,\ a'_1x_2 - a'_2x_1) = 0.$$

C_5 has a double point at (a'_1, a'_2, a'_3) and is of genus 5. The image of a line, $C_1 \equiv \sum_{k=1}^{3} a_k x_k = 0$, is the quintic

$$C'_5 \equiv \sum_{k=1}^{6} u_k(a_2x'_3 - a_3x'_2,\ a_3x'_1 - a_1x'_3,\ a_1x'_2 - a_2x'_1) v'_k(x') = 0.$$

C'_5 has a triple point at (a_1, a_2, a_3) and is of genus 3.

6. *Branchpoint and Coincidence Curves.*—The equation of $L(x)$ is the condition that the line and conic of (x') be tangent. Writing the equation of a tangent to (2) at (x'_1, x'_2, x'_3) and equating the coefficients of this equation to those of (1), we have

$$\sum_{k=1}^{6} u_k \frac{\partial v'_k}{\partial x'_i} = \rho x_i, \qquad i = 1, 2, 3. \tag{3}$$

Let $\quad v'_k \equiv a'_k x'^2_1 + b'_k x'^2_2 + c'_k x'^2_3 + 2d'_k x'_2 x'_3 + 2e'_k x'_3 x'_1 + 2f'_k x'_1 x'_2 = 0.$

Eliminating the x'_i and ρ from the three equations of (3) and from (1), we have the equation

$$L(x) \equiv \begin{vmatrix} \Sigma u_k a'_k & \Sigma u_k f'_k & \Sigma u_k e'_k & x_1 \\ \Sigma u_k f'_k & \Sigma u_k b'_k & \Sigma u_k d'_k & x_2 \\ \Sigma u_k e'_k & \Sigma u_k d'_k & \Sigma u_k c'_k & x_3 \\ x_1 & x_2 & x_3 & 0 \end{vmatrix} = 0.$$

$L(x)$ is of order 8 and genus 21 having no singularities. The image of L_8 is K'_{20}. It is of genus 21 and has 150 double points.

The equation of $L'(x')$ is the condition on the parameters x'_i that the line and cubic of (x) be tangent. It is the equation of the cubic in line coordinates. The coefficients in the equation of the line appear to degree equal to the class

of the cubic and the coefficients of the given curve appear to degree equal to that of the discriminant of a binary cubic.* The cubic is of class 6, the discriminant of order 4, and each element in it is of degree 2 in the x'_i. Hence $L'(x')$ is of order 14. L'_{14} has thirty-nine cusps and is of genus 39. The image of L'_{14} is K_{17} counted twice and the residual curve Γ_{36}. K_{17} and Γ_{36} are each of genus 39 and have 81 and 556 double points, respectively.

The order of $L(x)$ by Zeuthen's formula is $2(2+3-1)=8$. The order of $L'(x')$ is $2(3+5-1)=14$. This formula serves as a check when the equations of the branchpoint curves and the genera of the image curves are found by other methods.

K_{17} and Γ_{36} have thirty-nine contacts corresponding to the thirty-nine cusps of L'_{14}. The three image points corresponding to a point at a cusp of L'_{14} coincide at the corresponding contact of K_{17} and Γ_{36}. L'_{14} and K'_{20} have sixty-eight contacts corresponding to the contacts of L_8 and K_{17}, and 144 intersections corresponding to the contacts of L_8 and Γ_{36}.

7. *Successive Images of Lines.*—To a line $C'_1(x')$ corresponds $C_5(x)$ having one variable double point. The image of C_5 is C'_{25} which consists of C'_1 counted three times, and a residual C'_{22} which passes through the twenty intersections of C'_1 and K'_{20} corresponding to the twenty contacts of C_5 and L_8. The other two intersections of C'_{22} and C'_1 correspond to the variable double point of C'_5. C'_{22} and L'_{14} intersect in 308 points which consist of eighty-five contacts corresponding to the eighty-five intersections of C_5 and K_{17}, and 138 intersections corresponding to 138 of the 180 intersections of C_5 and Γ_{36}. To the remaining forty-two intersections of C_5 and Γ_{36} correspond the forty-two intersections of L'_{14} and C'_1 counted three times.

To a line $C_1(x)$ corresponds $C'_5(x')$ having one variable triple point. The image of C'_5 is C_{25} consisting of C_1 counted twice and a residual C_{23}. Of the twenty-three intersections of C_1 and C_{23}, six correspond to the triple point on C'_5 and the remaining seventeen are at the seventeen intersections of K_{17} and C_1 which correspond to the seventeen contacts of C'_5 and L'_{14}. To the remaining thirty-six intersections of C'_5 and L'_{14} correspond the thirty-six intersections of Γ_{36} and C_1, which are not on C_{23}. C_{23} and L_8 intersect in 184 points which are ninety-two contacts corresponding to ninety-two of the 100 intersections of C'_5 and K'_{20}. To the remaining eight intersections of C'_5 and K'_{20} correspond the sixteen intersections of C_1 counted twice and L_8.

Two lines C'_1, $\overline{C}'_1$ of (x') have C_5, $\overline{C}_5$ for images in (x) which intersect in twenty-five points. Three of these are collinear and correspond to the common

* See Salmon's "Higher Plane Curves," 4th ed., Art. 91, 188, 222.

point of C_1', $\overline{C}_1'$. The remaining twenty-two points have for their images the twenty-two intersections of C_1' with $\overline{C}_{22}'$ and $\overline{C}_1'$ with C_{22}'. The images of two lines C_1, $\overline{C}_1$ of (x) are C_5', $\overline{C}_5'$ of (x') which intersect in twenty-five points. Two of these intersections correspond to the common point of C_1, $\overline{C}_1$, and the remaining twenty-three have for their images the twenty-three intersections of C_1 with $\overline{C}_{23}$, and $\overline{C}_1$ with C_{23}, and also twenty-three intersections of C_{23} with $\overline{C}_{23}$.

8. *Cubics in* (x) *with One Basis Point.*—Let P_1 be a simple basis point on the cubics of (x). The image of C_1' is C_5 passing through P_1 and having a variable double point. The image of C_1 is C_5' which has a variable triple point. The image of P_1 is the fixed line f_1'. The image of f_1' is P_1 and a residual f_5 with a triple point at P_1. K_{17} and f_5 have the same three tangents at P_1.

$L(x)$ is of order 8 and genus 20, having a double point at P_1. Its proper image is K_{19}'. $L'(x')$ is of order 14 and genus 39 having thirty-nine cusps. K_{17} has a triple point and Γ_{36} an eight-fold point at P_1.

A line C_1 through P_1 corresponds to f_1' and C_4'. C_4' has a variable double point on f_1' and is of genus 2. The two simple intersections of f_1' and C_4' are images of the direction of C_1 through P_1. The image of C_4' is C_1 counted twice and a residual C_{18} with a double point at P_1. C_{18} meets C_1 in sixteen points apart from P_1, fourteen of which are at the non-basic intersections of C_1 and K_{17}, and two correspond to the variable double point of C_4'.

9. *Conics in* (x) *with One Basis Point.*—Let the simple basis point be P_1'. The image of C_1' is C_5 which has a variable double point. The image of C_1 is C_5' passing through P_1' and with a variable triple point. The image of P_1' is the fundamental line f_1. To f_1 corresponds P_1' and a residual f_5' with a four-fold point at P_1'. K_{20}' and f_5' have the same four tangents at P_1'. $L(x)$ is of order 8 and genus 21. K_{20}' has a four-fold point at P_1' and 144 double points. $L'(x')$ is of order 14 with a four-fold point at P_1'. It is of genus 36 and has thirty-six cusps. K_{16} and Γ_{34} have 66 and 487 non-basic double points, respectively.

To a line C_1' through P_1' corresponds f_1 and a C_4 of genus 3. Two of the intersections of f_1 and C_4 correspond to the direction of C_1' through P_1'. The image of C_4 is C_1' counted three times and a residual C_{17}' passing through P_1' simply. C_{17}' meets C_1' in sixteen points apart from P_1' which are at the sixteen non-basic intersections of K_{20}' and C_1'.

10. *Cubics in* (x) *with* $i \leqq 9$ *Basis Points and Conics in* (x') *with* $j \leqq 4$ *Basis Points.*—In general there may be any combination of i simple basis points, $i \leqq 9$, on the cubics of (x), and $j \leqq 4$ on the conics of (x'), and any par-

ticular case may be obtained from the following results by giving to i and j their appropriate values. Let the basis points of (x) be denoted by P_i and those of (x') by P'_j. The image of C'_1 is C_5 passing simply through each of the P_i of genus 5 and having a variable double point. The image of C_1 is C'_5 passing simply through each of the P'_j of genus 3 and having a variable triple point. To each basis point P_i corresponds a fixed line f'_i. To each f'_i corresponds the P_i whose image it is, and a residual f_5 with a triple point at that P_i and passing simply through the remaining $i-1\,P_i$. To each P'_j corresponds a fixed line f_j. The image of f_j is the P'_j to which it corresponds and a residual f'_5 having a four-fold point at that P'_j and passing simply through the the remaining $j-1\,P'_j$.

$L(x)$ is of order 8 and genus $21-i$, having a double point at each of the P_i. K'_{20-i} has each of the P'_j of multiplicity 4 and $\frac{1}{2}(i-20)(i-15)-12j$ non-basic double points. L'_{14} is of genus $39-3j$ having $39-3j$ cusps and a four-fold point at each of the P'_j. K_{17-j} has a triple point at each of the P_i and $\frac{1}{2}(j^2-25j-6i+162)$ double points. Γ_{36-2j} has an eight-fold point at each of the P_i and $2(j^2-33j-14i+278)$ double points. Apart from the P'_j, L'_{14} and K'_{20-i} have $68-3i-4j$ contacts and $144-8i-8j$ intersections corresponding to the non-basic contacts of L_8 with K_{17-j} and Γ_{36-2j}, respectively.

The images of a general line through one basis point of either plane are similar to those in the cases previously discussed. The lines joining two or more basis points will now be considered. Since the cubics and conics are not composite, there can be no more than three collinear basis points in (x) and two in (x'). The image of the line joining two basis points P_1, P_2 of (x) consists of f'_1, f'_2 and a fixed C'_3 through their intersection and through all the P'_j. Let three basis points P_1, P_2, P_3 be collinear. To the line joining them corresponds f'_1, f'_2, f'_3 all concurrent and a fixed C'_2 not through their common intersection, but through the P'_j. The image of the line joining two basis points P'_1, P'_2 of (x') consists of f_1, f_2 and a fixed C_3 not through their intersection, but through all the P_i.

11. *Line Pencil in Each Plane.*—Let the lines in (x) given by equation (1) form a pencil whose vertex $P\equiv(0,0,1)$ is not on the cubics. Then the lines of (x') also form a pencil whose vertex $P'\equiv(0,0,1)$ can not be on the conics. The image of C'_1 is C_5 with a double point at P and of genus 5. To C_1 corresponds C'_5 with a triple point at P' and of genus 3. The image of P' is the cubic f_3 whose image in (x') is P' and a residual f'_{15} with a twelve-fold point at P'. The image of P is the conic f'_2 to which corresponds P and a residual f_{10} with a six-fold point at P. To the line p' through P' corresponds

f_3 and the line p counted twice through P. To p corresponds f'_2 and p' counted three times. The rays of the two pencils are in (1, 1) correspondence. $L(x)$ is of order 8 and genus 20 having a double point at P. K'_{18} has a twelve-fold point at P' and fifty double points. L'_{14} has thirty cusps, a six-fold point at P' and is of genus 33. K_{14} has a six-fold point at P and thirty double points. Γ_{24} has a sixteen-fold point at P and 100 double points.

12. Now assume that both defining equations are satisfied by $x_2=0$, $x'_1=0$. The image of C'_1 is $x_2=0$ and C_4 of genus 3 through P. The image of C_1 is $x'_1=0$ and C'_4 with a double point at P' and of genus 2. To P' corresponds $x_2=0$ and a conic f_2 not through P. To P corresponds $x'_1=0$ and the line f'_1 not through P'. The image of $x_2=0$ is $x'^3_1=0$ and f'_1. The image of $x'_1=0$ is $x^2_2=0$ and f_2. L_8 has $x^2_2=0$ as a component, the proper curve being a sextic of genus 10 and not through P. K'_{18} consists of $x'^6_1=0$ and a proper C'_{12} with a six-fold point at P'. L'_{14} has $x'^4_1=0$ as a component and a proper C'_{10} with a double point at P', of genus 17 and having eighteen cusps. K_{14} consists of $x^4_2=0$ and a C_{10} with a double point at P. Γ_{24} consists of $x^8_2=0$ and a C_{16} with a six-fold point at P. The fixed components $x_2=0$, $x'_1=0$ are illustrations of the curves D_1, D_2 discussed by R. Baldus.*

13. When, in addition to the line pencils, there are $i \leqq 7$ basis points on the cubics of (x) and $j \leqq 3$ basis points of the conics of (x'), the image of C'_1 is C_5 with a double point at P and simple points at each of the P_i, and the image of C_1 is C'_5 with a triple point at P', and passing simply through each of the P'_j. L_8 has double points at P and at each of the P_i. K'_{18-i} has a $(12-i)$-fold point at P and triple points at each of the P'_j. L'_{14} has a six-fold point at P' and four-fold points at each of the P'_j. K_{14-j} has a $(6-j)$-fold point at P and double points at each of the P_i. Γ_{24-2j} has a $(16-2j)$-fold point at P and four-fold points at each of the P_i. The image of P' is f_3 through the P_i, but not through P. The image of P is f'_2 through the P'_j, but not through P'. To each P'_j corresponds a line through P, and to each P_i a line through P'.

14. *Geometric Depiction of Pencil Cases.*—All the cases in which the bilinear equation has but two terms can be visualized after the method of Marletta † by means of a quintic surface in ordinary space. Let $F=0$ be a surface of order 5 with a triple point at P_3 and a double point at P_2. Choose two planes (x) and (x') not through P_2 or P_3 as the double and triple plane, respectively, upon which the correspondence is to be pictured. Take any point P in (x) and any point P' in (x'). Connect P with P_3. The line PP_3 meets F in

* Baldus, *loc. cit.*, 2, Articles 6, 7, 8.

† G. Marletta, *loc. cit.*

two points not at P_3. Project these two points from P_2 upon (x') obtaining in (x') two points that are the images of the point P in (x). Going the other way, join P' with P_2 and project the three points of intersection of $P'P_2$ and F, not at P_2, on (x) from P_3. This gives in (x) the three images of P'.

The apparent contour of F from P_2, *i. e.*, the tangent cone, is cut by (x') in the curve L'_{14}. Every tangent line, elements of the tangent cone from P_2, meets F in one other point. The projections of the locus of the points of tangency and the locus of the residual intersections upon (x) from P_3 give the curves K_{14} and Γ_{24}, respectively. In like manner the tangent cone to F from P_3 is cut by (x) in L_8, and its curve of contact is projected upon (x') from P_2 into K'_{18}.

If $x_2=0$, $x'_1=0$ satisfy both defining equations, the surface F_5 consists of a plane through P_2P_3 and a residual surface F_4 passing through P_2 and having a double point at P_3. The depiction on (x) and (x') may now be obtained as before. Basis points in (x) and (x') are accounted for by fixed lines on F passing through P_2 and P_3 respectively.

15. *Type II.*—The defining equations of this type are obtained by interchanging the parameters of the line and conic of (x'). The image of a line, $C'_1 \equiv \sum_{k=1}^{3} a'_k x'_k = 0$, is,

$$C_7 \equiv \sum_{k=1}^{3} x_k v'_k (a'_2 u_3 - a'_3 u_2,\ a'_3 u_1 - a'_1 u_3,\ a'_1 u_2 - a'_2 u_1) = 0.$$

Since the three cubics

$$u_1/a'_1 = u_2/a'_2 = u_3/a'_3$$

form a pencil, their nine intersections are variable double points on C_7, leaving it of genus 6. The image of C_1 is C'_7 with four variable triple points at the common points of the pencil of conics whose parameters are those of C_1. C'_7 is of genus 3.

The equation of $L(x)$ is found as in Type I. L_8 is of genus 21 and has no singularities. K'_{28} is of genus 21 and has 330 double points. As in Type I we find that L'_{16} has fifty-seven cusps and is of genus 48. K_{23} and Γ_{66} have 183 and 2032 double points respectively.

16. *Type III.*—The defining equations are

$$\sum_{k=1}^{5} u_k(x) v'_k(x') = 0, \tag{1}$$

$$\sum_{k=1}^{3} v_k(x) x'_k = 0, \tag{2}$$

wherein $u_k(x)=0$, $v_k(x)=0$ are conics of (x) having a basis point at $R\equiv(1,0,0)$ and $v'_k(x')=0$ are general conics of (x'). To C'_1 corresponds C_6 with a triple point at R and three variable double points at the other three intersections of a pencil of conics whose parameters are those of C'_1. C_6 is of genus 4. The image of C_1 is C'_6 with seven variable double points and of genus 3. If C_1 passes through R its proper image is a cubic of genus 1. The image of of R is a fixed cubic of genus 1.

L_8 has a four-fold point at R and is of genus 15. K'_{18} has 121 double points. The equation of $L'(x')$ is the condition that two conics through a fixed point have a contact not at that point. This condition is of order 4 in the coefficients of each of the conics. Then $L'(x')$ is of order 12. It has twenty-seven cusps and is of genus 28. K_{20} has R of multiplicity 11, and eighty-eight double points. Γ_{32} has a fourteen-fold point at R and 346 double points.

17. *Type IV.*—The correspondence is defined by equations of the same form as those in Type III, wherein now $u_k(x)=0$, $v_k(x)=0$ are cubics through six simple basis points P_i of (x). The image of C'_1 is C_9 with each of the P_i three-fold, and with double points at the three variable intersections of a pencil of cubics whose parameters vary with C'_1. To C_1 corresponds C'_9 with twenty-three variable double points. If C_1 passes through one P_i its proper image is C'_6. To each P_i corresponds a fundamental cubic. If C_1 contains two P_i its proper image is a fixed C'_3. The proper image of a cubic through the six P_i is a C'_9.

L_{12} has a four-fold point at each P_i and is of genus 19. K'_{18} has 117 double points. The equation of $L'(x')$ is the condition that the two cubics through the P_i be tangent. This condition is of order 6 in the coefficients of each. L'_{18} has seventy-five cusps and is of genus 61. K_{33} has an eleven-fold point at each P_i and 105 double points. Γ_{96} has a thirty-two-fold point at each P_i and 1428 double points.

18. *Type V.*—The defining equations are written as in Type III. For the present type $u_k(x)=0$ represent cubics with a double point at $P\equiv(0,0,1)$ and a simple point at $R\equiv(1,0,0)$ and $v_k(x)=0$ are conics through P and R. The image of C'_1 is C_7 with P four-fold, R three-fold and two double points at the two variable intersections of a pencil of conics through P and R whose parameters vary with C'_1. C_7 is of genus 4.

In order to study the properties of the image of C_1, we may write the defining equations as follows:

$$\sum_{k=1}^{6} u_k(x)\,v'_k(x') \equiv x_2(a'x_1^2+b'x_2x_3+c'x_1x_3+e'x_1x_2+f'x_2^2)+d'x_1^2x_3=0, \qquad (1)$$

$$\sum_{k=1}^{3} v_k(x)\,x'_k \qquad \equiv x_2(p'x_1+q'x_2+r'x_3)+s'x_1x_3=0, \qquad (2)$$

wherein a', b', c', d', e', f' are quadratic and p', q', r', s' linear in the x'_i. Multiply (1) by s', (2) by $d'x_1$ and subtract, using the resulting equation

$$s'(a'x_1^2+b'x_2x_3+c'x_1x_3+e'x_1x_2+f'x_2^2)-d'x_1(p'x_1+q'x_2+r'x_3)=0, \qquad (3)$$

with (2) as the pair of defining equations. Eliminate the x_i from the equation of C_1 and (2) and (3) and we obtain a C'_8 with double points at the two intersections of $s'=0$, $d'=0$ and containing $s'=0$ as a fixed component. Then the proper image of C_1 is C'_7, which passes simply through the two basis points $s'=0$, $d'=0$. It is of genus 4 and has eleven variable double points.

The proper image of a line through P is a cubic which does not pass through $s'=0$, $d'=0$. The image of P is f'_4. The proper image of a line through R is a quartic not through $s'=0$, $d'=0$. To R corresponds f'_3. Both f'_4 and f'_3 pass through $s'=0$, $d'=0$. The image of the fundamental line PR is $f'_3f'_4$. The image curves of (x) intersect PR only at the basis points P and R. Every point of PR except P and R corresponds to the two basis points $s'=0$, $d'=0$. To each of the basis points $s'=0$, $d'=0$ corresponds the line PR. The image of $s'=0$ is PR counted twice, and a fixed C_5 with a triple point at P and a double point at R. The image of $d'=0$ is PR counted twice and a fixed C_{12}, with P seven-fold and R five-fold.

L_{10} has P six-fold and R four-fold and is of genus 15. K'_{17} has 105 double points. The condition that the cubic and conic of (x) defined by (1) and (2) be tangent is of order 4 in the coefficients of each, giving $L'(x')$ of order 12. L'_{12} has twenty-seven cusps and is of genus 28. Neither L'_{12} nor K'_{17} pass through the basis points $s'=0$, $d'=0$. K_{23} has a fourteen-fold point at P, a nine-fold point at R and seventy-six double points. Γ_{38} has a twenty-fold point at P, an eighteen-fold point at R and 295 double points.

§ 3. Class 2. Four Types.

19. *Type VI.*—The defining equations are

$$\sum_{k=1}^{3} x_k\psi'_k(x')=0, \qquad (1)$$

$$x'_1u_1(x)+x'_3u_3(x)=0, \qquad (2)$$

wherein $u_1(x)=0$, $u_3(x)=0$ are cubics of (x), and

$$\psi'_k(x')\equiv x_2'^2u'_k(x'_1,\, x'_3)+x'_2v'_k(x'_1,\, x'_3)+w'_k(x'_1,\, x'_3)=0,$$

(u'_k, v'_k, w'_k being homogeneous functions of x'_1, x'_3 of the respective degrees $n-2$, $n-1$, n) are curves of order n with $(n-2)$-fold points at $Q'\equiv(0,1,0)$, the vertex of the line pencil of (x'). The cubics of (x) would have no basis points but for the fact that the lines of (x') form a pencil. This forces the cubics of (x) to form a pencil, thus introducing nine simple basis points P_i into (x).

The image of C'_1 is C_{3n+1} with an n-fold point at each P_i and of genus $3n$. If C'_1 passes through Q' its proper image is a cubic counted twice through the P_i. The image of Q' is f_{3n-5} with an $(n-2)$-fold point at each P_i. To C_1 corresponds C'_{3n+1} with a $(3n-5)$-fold point at Q', and a triple point at each of the $4n-4$ variable intersections of a pencil of curves of order n whose parameters vary with C_1. C'_{3n+1} is of genus $3n-3$. If C_1 passes through a P_i, its proper image is C'_{2n+1} with Q' of multiplicity $2n-3$. If C_1 passes through two P_i its proper image is C'_{n+1} with an $(n-1)$-fold point at Q'. The image of each P_i is f'_i of order n with Q' $(n-2)$-fold. The proper image of a cubic through the P_i is a line counted three times through Q'. The line pencil of (x') is in $(1,1)$ correspondence with the pencil of cubics of (x).

To find the equation of $L(x)$, solve equation (2) for x'_3 in terms of x'_1, substitute this value of x'_3 in (1) and rearrange, obtaining a quadratic in x'_2/x'_3 whose discriminant equated to zero is $L(x)$. $L(x)$ is of order $6n-4$ and genus $12n-12$ with a $(2n-2)$-fold point at each P_i. $K'(x')$ is of order $6n-2$ with a $(6n-8)$-fold point at Q' and $18n-18$ double points. $L'(x')$, found as in Type I, is of order $6n+4$, genus $30n-15$, has Q' $(6n-8)$-fold and $36n-18$ cusps. $K(x)$ is of order $12n-12$ and has a $(4n-2)$-fold point at each P_i and $18n-6$ double points. $\Gamma(x)$ is of order $48n-32$ and has a $(16n-12)$-fold point at each P_i and $162n-126$ double points. Apart from Q', L'_{6n-4} and K'_{6n-2} have $24n-14$ contacts and $60n-44$ intersections corresponding to the non-basic contacts of L_{6n-4} with K_{12n-2} and Γ_{48n-32} respectively.

20. *Type VII.*—The defining equations are

$$\sum_{k=1}^{t}\phi_k(x)\psi'_k(x')=0, \tag{1}$$

$$x_1x'_1+x_2x'_3=0. \tag{2}$$

$$\phi_k(x)\equiv x_3^3u_k(x_1,\, x_2)+x_3^2v_k(x_1,\, x_2)+x_3w_k(x_1,\, x_2)+s_k(x_1,\, x_2)=0$$

(wherein u_k, v_k, w_k, s_k are homogeneous functions of the respective degrees $m-3$, $m-2$, $m-1$, m in x_1, x_2) is a C_m with an $(m-3)$-fold point at $P\equiv(0,0,1)$, the vertex of the line pencil of (x). $\psi'_k(x')$ is defined as in Type VI. Equation (1) has t terms where t takes the value $3n$ or $4m-2$

according as m is greater or less than $1/4(3n+2)$. A special case of this type for $n=2$, $m=3$ has been discussed in Article 11.

The image of C'_1 is C_{m+n} with an $(m+n-3)$-fold point at P and of genus $2(m+n)-5$. The image of C_1 is C'_{m+n} with an $(m+n-2)$-fold point at Q' and of genus $m+n-2$. The proper image of a line p through P is a line q' counted three times through Q', and the proper image of q' is p counted twice. The pencils of (x) and (x') are in $(1,1)$ correspondence. To P corresponds f'_{m+n-3} with Q' $(m+n-5)$-fold. To Q' corresponds f_{m+n-2} with an $(m+n-5)$ fold point at P.

$L(x)$, obtained as in Type VI, is of order $2(m+n)-2$ and genus $10(m+n)-30$, having a $[2(m+n)-8]$-fold point at P. $K'(x')$ is of order $6(m+n)-12$ having Q' of multiplicity $6(m+n)-18$ and $20(m+n)-50$ double points. To obtain the equation of $L'(x')$, solve equation (2) for x_2 in terms of x_1 and substitute this value of x_2 in (1). Then (1) may be written as a cubic in x_3/x_1 whose discriminant equated to zero is the equation of $L'(x')$. It is of order $4(m+n)-6$ and genus $16(m+n)-47$ with Q' of multiplicity $4(m+n)-14$ and $12(m+n)-30$ cusps. $K(x)$ is of order $4(m+n)-6$ with P of multiplicity $4(m+n)-14$ and $12(m+n)-30$ double points. $\Gamma(x)$ is of order $8(m+n)-16$ with P of multiplicity $8(m+n)-24$ and $40(m+n)-100$ double points.

21. *Notation.*—In the types of Classes 2 and 3 the defining equations represent respectively the same curve systems in the double plane as in Class 1. Thus the discussion of the succeeding types is so similar to the corresponding types of the first class that only a tabulation of the results is essential in most cases. The following notation will be used in this tabulation:

The symbol "$\sim$" meaning "corresponds to";

L, K', L', K, Γ, fixed curves as defined heretofore;

f, f', fundamental curves of (x) and (x');

C, C', variable curves of (x) and (x');

P, Q, R, P', Q', R', basis points of (x) and (x');

$\overline{P}$, non-basic but fixed points of either plane;

$\overline{\overline{P}}$, variable points of either plane;

p, genus of curve being described;

k, cusps of L';

Subscripts of curves denote their order;

Subscripts of points denote their multiplicity on the curve being described; (the subscripts i and j, however, denote any one of a given number of basis points or curves playing the same rôle).

The following types will illustrate the use of this notation:

22. *Type VIII.*—The defining equations are

$$\sum_{k=1}^{4} u_k(x)\,\psi'_k(x') \;=0, \tag{1}$$

$$x'_1 v_1(x) + x'_3 v_3(x) = 0. \tag{2}$$

in which $u_k(x)=0$, $v_1(x)=0$, $v_3(x)=0$, are cubics with six simple basis points P_i. The cubics of equation (2) form a pencil because their parameters belong to a line pencil, and thus introduce three additional basis points Q_j into (x). The pencil of cubics of (x) and the line pencil of (x') are in (1, 1) correspondence.

$C'_1 \sim C_{3n+3}$, $p=3n+1$; $6P_{n+1}$, $3Q_n$.
$C_1 \sim C'_{3n+3}$, $p=3n-1$; Q'_{3n-3}, $(12n-4)\overline{\overline{P}}_2$.
$P_i \sim f'_{i,\,n+1}$; Q'_{n-1}.
$Q_j \sim f'_{j,\,n}$; Q'_{n-2}.
$Q' \sim f_{3n-3}$; $6P_{n-1}$, $3Q_{n-2}$.
L_{6n}, $p=12n-8$; $6P_{2n}$, $3Q_{2n-2}$.
K'_{6n}, $p=12n-8$; Q'_{6n-6}, $(18n-12)\overline{P}_2$.
L'_{6n+6}, $p=30n-15$; Q'_{6n-6}, $(36n-6)k$.
K_{12n+6}, $p=30n-15$; $6P_{4n+2}$, $3Q_{4n-2}$, $18n\overline{P}_2$.
Γ_{48n-12}, $p=30n-15$; $6P_{16n-4}$, $3Q_{16n-8}$, $(167n-72)\overline{P}_2$.

23. *Type IX.*—The defining equations are written as in Type VIII. For the present type, $v_1(x)=0$, $v_3(x)=0$ represent cubics of (x) passing through eight basis points P_i and therefore determining a ninth Q; $u_k(x)=0$ represents curves of order 9 with triple points at each of the eight P_i, but not passing through Q.

$C'_1 \sim C_{3n+9}$, $p=3n+4$; $8P_{n+3}$, Q_n.
$C_1 \sim C'_{3n+9}$, $p=3n+5$; Q'_{3n+3}, $(12n+20)\overline{\overline{P}}_2$.
$Q' \sim f_{3n+3}$; $8P_{n+1}$, Q_{n-2}.
$P_i \sim f'_{i,\,n+3}$; Q'_{n+1}.
$Q \sim f'_n$; Q'_{n-2}.
L_{6n+12}, $p=12n+4$; $8P_{2n+4}$, Q_{2n-2}.
K'_{6n+6}, $p=12n+4$; Q'_{6n}, $(18n+6)\overline{P}_2$.
L'_{6n+12}, $p=30n+25$; Q'_{6n}, $(36n+30)k$,
K_{12n+30}, $p=30n+25$; $8P_{4n+10}$, Q_{4n-2}, $(18n+18)\overline{P}_2$.
Γ_{48n+48}, $p=30n+25$; $8P_{16n+16}$, Q_{16n+4}, $(162n+90)\overline{P}_2$.

§ 4. Class 3. Three Types.

24. *Type X.*—The defining equations are

$$\sum_{k=1}^{4} u_k(x)\,c_k'(x')=0, \tag{1}$$

$$\sum_{k=1}^{3} x_k v_k'(x')=0, \tag{2}$$

wherein $c_k'(x')=0$, $v_k'(x')=0$ are conics of (x') with basis points $Q'\equiv(0,1,0)$ and $R'\equiv(1,0,0)$ and $u_k(x)=0$ are general cubics of (x). The image of C_1 is C_8' with four-fold points at Q' and R' and triple points at the two variable intersections of a pencil of conics whose parameters depend on those of C_1. C_8' is of genus 3. To obtain the image of C_1' we may write the defining equations as follows:

$$ax_3'^2+bx_2'x_3'+cx_1'x_3'+dx_1'x_2'=0, \tag{1}$$

$$px_3'^2+qx_2'x_3'+rx_1'x_3'+sx_1'x_2'=0, \tag{2}$$

wherein a, b, c, d and p, q, r, s are respectively general cubic and linear functions of x_1, x_2, x_3. Multiply (1) by s, (2) by d, subtract (2) from (1), remove the common factor x_3 and we obtain the equation

$$(cs-rd)x_1'+(bs-qd)x_2'+(as-pd)x_3'=0. \tag{3}$$

Using (3) with either (1) or (2) as defining equations, the image of C_1' is found to be C_9 with double points at the sixteen intersections of a pencil of quartics whose parameters vary with C_1'. Three of these double points lie at the intersections of $s=0$, $d=0$, and the remaining thirteen are variable. The line $s=0$ is a fixed component of C_9 so the proper image of C_1' is C_8 passing through the three basis points $s=0$, $d=0$ and having thirteen variable double points. C_8 is of genus 8.

L_8, $p=21$.
K_{32}', $p=21$; Q_{16}', R_{16}', $(204)\overline{P}_2$.
L_{20}', $p=39$; Q_{10}', R_{10}', $42k$.
K_{18}, $p=39$; $97\overline{P}_2$.
Γ_{44}, $p=39$; $864\overline{P}_2$.

25. The following statements hold for all three types of Class 3. The images of Q' and R' are respectively the basis curves

$$bs-dq=0, \quad cs-rd=0.$$

The line $Q'R'$ is the image of each of the three basis points $s=0$, $d=0$ of (x). All the curves of (x') intersect $Q'R'$ only in points Q' and R' which are always points of equal multiplicity. There is no proper image of the line $Q'R'$, but

every point of it except Q' and R' corresponds to the three basis points $s=0$, $d=0$. The equation of $L(x)$ is the condition that the two conics of (x') be tangent, which is of order 2 in the coefficients of each. The curves $L(x)$, $K(x)$, $\Gamma(x)$ do not pass through the basis points $s=0$, $d=0$.

26. *Type XI.*—The defining equations are

$$\sum_{k=1}^{4} u_k(x)\, c'_k(x') = 0, \tag{1}$$

$$\sum_{k=1}^{4} v_k(x)\, v'_k(x') = 0, \tag{2}$$

wherein $u_k(x)=0$, $v_k(x)=0$ are conics with one basis point P.

$$\begin{aligned}
&C'_1 \sim C_8,\ p=6;\, P_4,\ (s=0,\ d=0)_1,\ 9\overline{\overline{P}}_2.\\
&C_1 \sim C'_8,\ p=3;\, Q'_4,\ R'_4,\ 6\overline{\overline{P}}_2.\\
&Q' \sim f_4;\, P_2,\ (s=0,\ d=0)_1.\\
&R' \sim f_4;\, P_2,\ (s=0,\ d=0)_1.\\
&P \sim f'_4;\, Q'_2,\ R'_2.\\
&L_8,\ p=15;\, P_4.\\
&K'_{24},\ p=15;\, Q'_{12},\ R'_{12},\ 106\overline{P}_2.\\
&L'_{16},\ p=25;\, Q'_8,\ R'_8,\ 24k.\\
&K_{18},\ p=25;\, P_{10},\ 66\overline{P}_2.\\
&\Gamma_{28},\ p=25;\, P_{12},\ 260\overline{P}_2.
\end{aligned}$$

27. *Type XII.*—In the defining equations, $u_k(x)=0$, $v_k(x)=0$ are cubics with six simple basis points P_i.

$$\begin{aligned}
&C'_1 \sim C_{12},\ p=10;\, 6P_4,\ (s=0,\ d=0)_1,\ 9\overline{\overline{P}}_2.\\
&C_1 \sim C'_{12},\ p=5;\, Q'_6,\ R'_6,\ 20\overline{\overline{P}}_2.\\
&P_i \sim f'_{i,4};\, Q'_2,\ R'_2.\\
&Q' \sim f_6;\, 6P_2,\ (s=0,\ d=0)_1.\\
&R' \sim f_6;\, 6P_2,\ (s=0,\ d=0)_1.\\
&L_{12},\ p=19;\, 6P_4.\\
&K'_{24},\ p=19;\, Q'_{12},\ R'_{12},\ 102\overline{P}_2.\\
&L'_{24},\ p=55;\, Q'_{12},\ R'_{12},\ 66k.\\
&K_{30},\ p=55;\, 6P_{10},\ 81\overline{P}_2.\\
&\Gamma_{84},\ p=55;\, 6P_{28},\ 1080\overline{P}_2.
\end{aligned}$$

§ 5. COMPLETENESS OF THE CLASSIFICATION.

28. *Reduction of Certain Cases to Type Form.*—It remains now to be proved that all cases of general (2, 3) point correspondences are birationally equivalent to certain of these twelve independent types. All cases where the

defining equations are made up of combinations of the curve systems used in the twelve types (except those that define (2, 3) compound involutions) either belong to these types or can be reduced to certain of them by quadratic transformations. These curve systems are as follows: In (x') lines and conics; line pencil and curves of order n with the vertex of the pencil $(n-2)$-fold; conics with two basis points. In (x) lines and cubics; line pencil and curves of order m with the vertex of the pencil $(m-3)$-fold; conics with one basis point; cubics with six basis points; cubics with a double and a simple basis point through which pass conics; cubics with eight basis points and curves of order 9 with triple points at each of them.

In Class 1, all the cases in which the lines of (x') form a pencil are particular cases of the corresponding types of Class 2 for $n=2$. When the conics of (x') form a pencil, they may be transformed into a line pencil by quadric inversion, by which also the lines of (x') are transformed into conics with three basis points. These cases are then birationally equivalent to particular cases of Class 2. All the above cases would be independent types of Class 1 if they were not forced, as it were, into special cases of types of Class 2 by the fact that the lines or conics of (x') can have but two homogeneous parameters because these parameters represent a pencil in (x). If in Type V the parameters of the line and conic are interchanged, the cubics of (x), since their parameters are now lines of (x'), can have but three homogeneous parameters and therefore have two basis points besides those required by the system in (x). By quadric inversion these cubics go into conics through the double point and the two added basis points. The conics of (x) through the two required basis points are invariant. Then the conics, transforms of the cubics, and the invariant conics have one point in common, so this case reduces to a special case of Type III.

In Class 2 if the system in (x) consists of lines and cubics and the lines are forced into a pencil by having the line pencil of (x') for parameters, we have a special case of Type VII for $m=3$. When the system of (x) is composed of conics with a common basis point, the conics having the line pencil of (x') as parameters form a pencil which goes into a pencil of lines by quadric inversion. By the same process the conics in the other equation are transformed into cubics none of whose basis points need be at the vertex of the line pencil. This case, then, is birationally equivalent to Type VII for $m=3$. When the system in (x) consists of cubics with a double and a simple point and conics through these points, if the conics form a pencil they correspond to a line pencil; the cubics are invariant under quadric inversion

so this also reduces to a special case of Type VII. When the parameters are interchanged the cubic pencil, since it contains a double point is transformed into a pencil of conics and thence into a pencil of lines. The conics given by the other equation are transformed into cubics with three basis points, none of which are at the vertex of the line pencil. This is also a special case of Type VII.

In Class 3 when the conics of (x') form a pencil in one equation we have a particular case of the type of Class 2 having the same system of curves in (x). When the system in (x) consists of cubics with a double and a simple point and conics through them, the cubics are given two more basis points because the conics of (x') allow but four homogeneous parameters in the equation. The cubics then reduce to conics through two points one of which is the double point, and as the conics given by the other equation may remain invariant, this case is birationally equivalent to a special case of Type IX.

If, in any combination of the curve systems used in the twelve types, the associated parameters so restrict each other that the curves of each of the two defining equations become pencils in their respective planes, a (2, 3) compound involution is established.

29. *Reduction of All Other Cases.*—Since any combination of the foregoing curve systems of (x) and (x') that has at least three parameters in one of its defining equations is birationally equivalent to some one of the twelve independent types, in order to prove the classification complete it remains only to show that any curve system having two [three] variable intersections is birationally equivalent to some one of the above curve system of (x') [(x)]. The proof of this is the same as that for the curve systems of the simple planes of the (1, 2) and (1, 3) point correspondences. From the inequalities connecting intersections of curves of such systems, it follows by reasoning exactly like that used by Bertini* that any given non-involutorial curve system intersecting in two [three] points can be reduced by a series of quadratic transformations to one of the preceding systems of (x') [(x)]. Any algebraic method of establishing a general (2, 3) point correspondence between two planes is therefore birationally equivalent to some one of the given twelve independent types.

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* E. Bertini, "Ricerche sulle trasformazioni univoche involutorie nel piano," *Annali di Matematica*, Ser. 2, Vol. VIII (1877), pp. 244—286.